XIII. *Hydraulic Investigations, subservient to an intended Croonian Lecture on the Motion of the Blood.* By Thomas Young, *M. D. For. Sec. R. S.*

Read May 5, 1808.

I. *Of the Friction and Discharge of Fluids running in Pipes, and of the Velocity of Rivers.*

HAVING lately fixed on the discussion of the nature of inflammation, for the subject of an academical exercise, I found it necessary to examine attentively the mechanical principles of the circulation of the blood, and to investigate minutely and comprehensively the motion of fluids in pipes, as affected by friction, the resistance occasioned by flexure, the laws of the propagation of an impulse through the fluid contained in an elastic tube, the magnitude of a pulsation in different parts of a conical vessel, and the effect of a contraction advancing progressively through the length of a given canal. The physiological application of the results of these inquiries I shall have the honour of laying before the Royal Society at a future time; but I have thought it advisable to communicate, in a separate paper, such conclusions, as may be interesting to some persons, who do not concern themselves with disquisitions of a physiological nature; and I imagine it may be as agreeable to the Society that they should be submitted at present to their consideration, as that they should be withheld until the time appointed for the delivery of the Croonian Lecture.

It has been observed by the late Professor ROBISON, that the comparison of the Chevalier DUBUAT's calculations with his experiments is in all respects extremely satisfactory; that it exhibits a beautiful specimen of the means of expressing the general result of an extensive series of observations in an analytical formula, and that it does honour to the penetration, skill, and address of Mr. DUBUAT, and of Mr. DE ST. HONORE', who assisted him in the construction of his expressions. I am by no means disposed to dissent from this encomium; and I agree with Professor ROBISON, and with all other late authors on hydraulics, in applauding the unusually accurate coincidence between these theorems and the experiments from which they were deduced. But I have already taken the liberty of remarking, in my lecture on the history of hydraulics, that the form of these expressions is by no means so convenient for practice as it might have been rendered; and they are also liable to still greater objections in particular cases, since, when the pipe is either extremely narrow, or extremely long, they become completely erroneous: for notwithstanding Mr. DUBUAT seems to be of opinion, that a canal may have a finite inclination, and yet the water contained in it may remain perfectly at rest, and that no force can be sufficient to make water flow in any finite quantity through a tube less than one twenty-fifth of an inch in diameter, it can scarcely require an argument to show that he is mistaken in both these respects. It was therefore necessary for my purpose to substitute, for the formulæ of Mr. DUBUAT, others of a totally different nature; and I could follow DUBUAT in nothing but in his general mode of considering a part of the pressure, or of the height of a given reservoir, as employed in overcoming the friction of the pipe through which the water flows out of it; a principle, which, if not of his original invention, was certainly first reduced by him into a practical form. By comparing the experiments, which he has collected, with some of GERSTNER, and some of my own, I have ultimately discovered a formula, which appears to agree fully as well as DUBUAT'S, with the experiments from which his rules were deduced, which accords better with GERSTNER'S experiments, which extends to all the extreme cases with equal accuracy, which seems to represent more simply the actual operation of the forces concerned, and which is direct in its application to practice, without the necessity of any successive approximations.

I began by examining the velocities of the water, discharged, through pipes of a given diameter, with different degrees of pressure; and I found that the friction could not be represented by any single power of the velocity, although it frequently approached to the proportion of that power, of which the exponent is 1.8; but that it appeared to consist of two parts, the one varying simply as the velocity, the other as its square. The proportion of these parts to each other must however be considered as different, in pipes of different diameters, the first part being less perceptible in very large pipes, or in rivers, but becoming greater than the second in very minute tubes, while the second also becomes greater, for each given portion of the internal surface of the pipe, as the diameter is diminished.

If we express, in the first place, all the measures in French inches, calling the height employed in overcoming the friction f, the velocity in a second v, the diameter of the pipe d, and its length l, we may make $f = a\frac{l}{d}v^2 + 2c\frac{l}{d}v$; for it is obvious that the friction must be directly as the length of the pipe; and since the pressure is proportional to the area of the section, and the surface producing the friction to its circumference or diameter, the relative magnitude of the friction must also be inversely as the diameter, or nearly so, as DUBUAT has justly observed. We shall then find that a must be $.0000001\left(480+\frac{75}{d}-\frac{1440}{d+12}-\frac{180}{d+\frac{1}{3}}\right)$, and $c=.0000001\left(\frac{900dd}{dd+1000}+\frac{1}{\sqrt{d}}\left(1050+\frac{12}{d}+\frac{.9}{dd}\right)\right)$. Hence it is easy to calculate the velocity for any given pipe or river, and with any given head of water. For the height required for producing the velocity, independently of friction, is, according to DUBUAT, $\frac{v^2}{478}$, or rather, as it appears from almost all the experiments which I have compared, $\frac{v^2}{550}$; and the whole height h is therefore equal to $f+\frac{v^2}{550}$, or $h=\left(\frac{al}{d}+\frac{1}{550}\right)v^2+\frac{2cl}{d}v$; and making $b=\frac{1}{al:d+.00182}$, and $e=\frac{bcl}{d}$, $v^2+2ev=bh$, whence $v=\sqrt{(bh+e^2)}-e$. In order to adapt this formula to the case of rivers, we must make l infinite; then b becomes $\frac{d}{al}$, and $bh=\frac{d}{a}.\frac{h}{l}=\frac{ds}{a}$, s being the sine of the inclination, and d four times the hydraulic mean depth; and since e is here $=\frac{c}{a}$, $v=\frac{\sqrt{(ads+cc)}-c}{a}$, and in most rivers, v becomes nearly $\sqrt{(20000\,ds)}$.

In order to show the agreement of these formulæ with the result of observation, I have extracted, as indiscriminately and impartially as possible, forty of the experiments made and collected by DUBUAT; I have added to these some of GERSTNER's, with a few of my own; and I have compared the results of these experiments with DUBUAT's calculations, and with my own formulæ, in separate columns. There are six of DUBUAT's experiments which he has rejected as irregular, apparently without any very sufficient reason, since he has accidently mentioned that some of them were made with great care: I have therefore calculated the velocities for these experiments in both ways, and compared the results in a separate table.

Tabular Comparison of Hydraulic Experiments.

Observer.	*d.*	1/*s*	Superf. Veloc.	*v.*	Dub.	Log. ratio.	Y.	Log. ratio.	*a* .17 ×	*c* 17 ×	√ 0000*ds*)
DUBUAT.	262.5	35723	15.96	12.56?	10.53	.0776	11.10	.0537	424	952	11.1
	258.5	6413	31.77	26.63?	28.76	.0334	28.02	.0221	424	952	28.3
	92.4	21827	9.61	7.01?	8.38	.0775	8.14	.0649	415	914	9.3
	75.6	27648	7.27	5.07?	6.55	.1112	6.27	.0923	413	887	7.5
	17.6	9288		5.70	5.86	.0120	5.97	.0291	376	465	6.1
	16.4	432		32.52	31.61	.0124	30.67	.0255	374	451	27.6
	11.7	1412		14.17	13.59	.0182	14.05	.0037	360	416	12.2
	9.9	427		22.37	24.37	.0372	24.41	0379	355	414	21.7
	5.8	212		27.51	27.19	.0051	27.34	.0027	332	466	23.5

Observers.	*d.*	*l.*	*h.*	*v.*	Dub.	Log. ratio.	Y.	Log. ratio.	*a* .17 ×	.17 ×
COUPLET	18	43200	145.08	39.16	40.51	.0148	38.49	.0075	376	469
	5	84240	25.00	5.32	5.29	.0024	5.40	.0065	326	492
			16.75	4.13	4.23	.0103	4.21	.0083		
			5.58	2.01	2.25	.0490	2.01	.0000		
BOSSUT	2.01	2160	24	24.73	24.08	.0115	24.76	.0006	287	747
			12	16.38	16.10	.0075	16 86	.0125		
		1080	24	35.77	35.10	.0082	35 05	.0089		
		360	24	58.90	58.80	.0007	56.85	.0154		
	1.33	2160	12	12.56	12.75	0065	13 28	0242	270	919
		1080	24	28 08	28.21	.0020	28.84	0110		
		360	24	48.53	49.52	.0088	48.66	0015		
	1.	600	12	22.28	21.98	.0055	22.83	0106	259	1063
			4	12 22	11.76	.0167	11.92	.0108		
DUBUAT		737	23.7	28.67	29.41	.0111	30 11	0213		
			12.2	19 99	19.95	.0009	20 67	.0145		
			4.2	10.56	10.66	.0041	10 90	0137		
		117	36	84 95	85.52	.0029	83 12	.0069		
			18	58.31	58.47	.0014	58.41	.0012		
	.24167	36.25	53.25	85 77	85.20	.0029	85.71	.0003	309	2268
			41.25	73.81	73.90	.0005	74 67	.0050		
			20.17	51 96	50.14	.0155	50 87	.0093		
			5.00	23.40	23.19	.0039	23.09	0058		
			.83	7.58	8.22	.0420	7.22	.0212		
	.1667	36.25	51.25	64.37	64.95	.0031	64 08	.0021	402	2827
			38.75	54.19	55.32	.0090	54 93	0055		
			15.29	33.38	33.17	.0028	32.67	.0094		
			2.04	10 62	10.49	.0053	9 24	0604		
	.125	34.17	42.17	45.47	46.21	.0070	45 88	.0039	518	3405
			35.33	41.61	41.71	.0010	41 55	.0006		
			14.58	26.20	25.52	.0114	24.94	.0214		
			2.08	7.32	8 35	.0572	6.98	.0206		

(Mean .0178 (Mean .0169
= L. 1.042) = L. 1.040)

Observers.	*d.*	*l.*	*b.*	*v.*	Dub.	Log. rat.	Y.	Log. rat.	*a.*	*c.*
GERSTNER, at 55.5°F.	.2	63	10.7	24.2	23.9	.006	24.1	.002	349	2533
			7.7	21.0	19.9	.023	19.1	.042		
			4.7	15.8	14.9	.026	13.9	.056		
			1.7	7.5	8.2	.039	6.9	.036		
			.7	2.5	5.0	.301	3.4	.133		
	.133	33	10.7	27.1	23.4	.064	22.5	.081	488	3259
			7.7	23.2	19.4	.077	18.5	.098		
			4.7	15.4	14.6	.024	13.5	.058		
			1.7	5.6	8.1	.160	6.7	.078		
			.7	2.3	4.6	.301	3.4	.169		
	.0674	33	10.7	10.0	8.9	.051	10.1	.004	975	5700
			7.7	7.2	7.4	.012	8.2	.057		
			4.7	4.5	5.6	.095	5.6	.095		
			1.7	1.5	3.1	.316	2.5	.222		
			.7	.5	1.8	.444	1.1	.342		
						(Mean .129=L.1.346		.098=L.1.254)		
Y. At 60°.	$\frac{1}{42}$	8.50	32.4	14 40	0	∞	13.36	.032	2956	13882
	$\frac{1}{183}$	3.42	30.0	.53			.52	.008	13404	452100
		1.17	5.8	.27			.30	.046		
								(Mean .029=L.1.068)		
DUBUAT	2	255.25	36.35	86.31	84.2	.011	79.7	.035	287	747
	1	24	36.25	122.59	117.8	.018	120.8	.007	259	1063
			27	106.45	101.1	.022	104.1	.010		
			18	84.85	82.2	.013	84.8	.000		
			9	59.25	57.5	.013	59.7	.004		
		4	27.08	118.67	111.5	.027	118.5	.000		
						(Mean .017=L.1.041		.009=L.1.022)		

It appears from this comparison, that in the forty experiments extracted from the collection, which served as a basis for DUBUAT's calculations, the mean error of his formula is $\frac{1}{24}$ of the whole velocity, and that of mine $\frac{1}{25}$ only; but if we omit the four experiments, in which the superficial velocity only of a river was observed, and in which I have calculated the mean velocity by DUBUAT's rules, the mean error of the remaining 36 is $\frac{1}{35}$, according to my mode of calculation, and $\frac{1}{37}$ according to Mr. DUBUAT'S; so that on the whole, the accuracy of the two formulæ may be considered as precisely equal with respect to these experiments. In the six experiments which DUBUAT has wholly rejected, the mean error of his formula is about $\frac{1}{24}$, and that of mine $\frac{1}{45}$. In fifteen of GERSTNER'S experiments, the mean error of DUBUAT'S rule is one third, that of mine one fourth; and in the three experiments which I made with very fine tubes, the error of my own rules is one fifteenth of the whole, while in such cases DUBUAT'S formulæ completely fail. I have determined the mean error by adding together the logarithmic ratios of all the results, and dividing the sum by the number of experiments. It would be useless to seek for a much greater degree of accuracy, unless it were probable that the errors of the experiments themselves were less than those of the calculations; but if a sufficient number of extremely accurate and frequently repeated experiments could be obtained, it would be very possible to adapt my formula still more correctly to their results.

In order to facilitate the computation, I have made a table of the coefficients *a* and *c* for the different values of *d*, all the measures being still expressed in French inches.

Table of Coefficients for French Inches.

d	a $.1^7\times$	c $.1^7\times$	d	a $.1^7\times$	c $.1^7\times$	d	a $.1^7\times$	c $.1^7\times$	d	a $.1^7\times$	c $.1^7\times$
∞	430	900	40	400	719	4	319	540	.4	257	1717
500	427	943	30	393	618	3	305	617	$\frac{1}{3}$	268	1895
400	426	946	25	387	560	2.5	296	687	.3	279	2008
300	423	950	20	380	492	2	288	751	$\frac{1}{4}$	303	2225
200	421	951	15	370	427	1.5	275	866	.2	349	2532
100	416	923	10	354	414	1	259	1063	$\frac{1}{6}$	402	2827
90	415	911	9	350	421	.9	255	1123	.15	440	3026
80	413	896	8	345	433	.8	252	1193	$\frac{1}{7}$	458	3116
70	410	872	7	340	440	.7	249	1278	$\frac{1}{8}$	518	3405
60	408	840	6	335	462	.6	248	1384	$\frac{1}{9}$	589	3693
50	406	792	5	325	512	.5	249	1524	.1	646	3985

For example, in the last experiment, where d is 1, l 4, and h 27.1, we have $a = .0000259$, $b = \frac{1}{al:d + .00182} = 516$, $c = .0001063$, $e = bcl:d = .22$, and $v = \sqrt{(bh + e^2)} - e = 118.46$, which agrees with the experiment within $\frac{1}{500}$ of the whole. I had at first employed for a the formula $\frac{430}{1 + 1.2:d} + \frac{57}{d} + \frac{1}{6dd}$, but I found that the value, thus determined, became too great when d was about 20, and too small in some other cases. Coulomb's experiments on the friction of fluids, made by means of the torsion of wires, give about .00014 for the value of c, which agrees as nearly with this table, as any constant number could be expected to do. I have however reason to think, from some experiments communicated to me by Mr. Robertson Buchanan, that the value of a, for pipes about half an inch in diameter, is somewhat too small; my mode of calculation, as well as Dubuat's, giving too great a velocity in such cases.

If any person should be desirous of making use of Dubuat's formula, it would still be a great convenience to begin by determining v according to this method; then, taking $b = \frac{l}{b - v^2 : 478}$, or rather, as LANGSDORF makes it, $b = \frac{l}{b - v^2 : 482}$, to proceed in calculating v by the formula $v = 148.5$ $(\sqrt{d} - .2) \cdot \left(\frac{1}{\sqrt{b} - \text{H. L.} \sqrt{(b + 1.6)}} - .001 \right)$, since this determination of b will, in general, be far more accurate than the simple expression $b = \frac{l + 45d}{b}$, and the continued repetition of the calculation, with approximate values of v, may thus be avoided. Sometimes, indeed, the values of v found by this repetition, will constitute a diverging instead of a converging series, and in such cases, we can only employ a conjectural value of v, intermediate between the two preceding ones.

Having sufficiently examined the accuracy of my formula, I shall now reduce it into English inches, and shall add a second table of the coefficients, for assisting the calculation. In this case, a becomes $.0000001 \left(413 + \frac{75}{d} - \frac{1440}{d + 12.8} - \frac{180}{d + .355} \right)$, $c = .0000001 \left(\frac{900dd}{dd + 1136} + \frac{1}{\sqrt{d}} \left(1085 + \frac{13.21}{d} + \frac{1.0563}{dd} \right) \right)$, and $b = \frac{1}{al : d + .00171}$, e being $\frac{bcl}{d}$, and $v = \sqrt{(bh + e^2)} - e$, or $= \sqrt{\left(\frac{ds}{a} + \frac{cc}{aa} \right)} - \frac{c}{a}$, as before; and in either case the superficial velocity of a river may be found, very nearly, by adding to the mean velocity v its square root, and the velocity at the bottom by subtracting it.

Table of Coefficients, for English Inches.

d.	*a.* .17 ×	*c.* .17 ×	*d.*	*a.* .17 ×	*c.* .17 ×	*d.*	*a.* .17 ×	*c.* .17 ×	*d.*	*a.* .17 ×	*c.* .17 ×
∞	413	900	40	383	698	4	306	556	.4	254	1779
500	410	944	30	377	597	3	292	635	$\frac{1}{3}$	268	1963
400	409	948	25	371	526	2.5	284	694	.3	280	2082
300	406	951	20	364	482	2	277	774	$\frac{1}{4}$	305	2307
200	404	951	15	354	430	1.5	266	894	.2	354	2631
100	399	918	10	339	413	1	251	1099	$\frac{1}{6}$	409	2943
90	398	903	9	336	421	.9	248	1161	.15	447	3150
80	396	885	8	331	433	.8	245	1234	$\frac{1}{7}$	466	3251
70	393	860	7	327	449	.7	243	1322	$\frac{1}{8}$	528	3558
60	391	825	6	322	471	.6	243	1433	$\frac{1}{9}$	599	3866
50	389	772	5	312	507	.5	245	1578	.1	657	4183

II. *Of the Resistance occasioned by Flexure in Pipes or Rivers.*

Mr. DUBUAT has made some experiments on the effect of the flexure of a pipe in retarding the motion of the water flowing through it; but they do not appear to be by any means sufficient to authorise the conclusions which he has drawn from them. He directs the squares of the sines of the angles of flexure to be collected into one sum, which, being multiplied by a certain constant coefficient, and by the square of the velocity, is to show the height required for overcoming the resistance. It is, however, easy to see that such a rule must be fundamentally erroneous, and its coincidence with some experiments merely accidental, since the results afforded by it must vary according to the method of stating the problem, which is entirely arbitrary. Thus it depended only on Mr. DUBUAT to consider a pipe bent to an angle of 144° as consisting of a single flexure, as composed of two flexures of 72° each, or of a much greater number of smaller flexures, although the result of the experiment would only agree with the arbitrary division into two parts, which he has adopted. This difficulty is attached to every mode of computing the effect either from the squares of the sines or from the sines themselves; and the only way of avoiding it is to attend merely to the angle of flexure as expressed in degrees. It is natural to suppose that the effect of the curvature must increase, as the curvature itself increases, and that the retardation must be inversely proportional to the radius of curvature, or very nearly so; and this supposition is sufficiently confirmed, by the experiments, which Mr. DUBUAT has employed in support of a theory so different. It might be expected that an equal curvature would create a greater resistance in a larger pipe than in a smaller, since the inequality in the motions of the different parts of the fluid is greater; but this circumstance does not seem to have influenced the results of the experiments made with pipes of an inch and of two inches diameter: there must also be some deviation from the general law, in cases of very small pipes having a great curvature, but this deviation cannot be determined without further experiments. Of the 25 which DUBUAT has made, he has rejected 10 as irregular, because they do not agree with his theory: indeed 4 of them, which were made with a much shorter pipe than the rest, differ so manifestly from them that they cannot be reconciled: but 5 others agree sufficiently, as well as all the rest, with the theory which I have here proposed, supposing the resistance to be as the angular flexure, and to increase besides almost in the same proportion as the radius of curvature diminishes, but more nearly as that power of the radius of which the index is $\frac{7}{8}$. Thus if p be the number of degrees subtended at the centre of flexure, and q the radius of curvature of the axis of the pipe in French inches, we shall have $r = \frac{pv^2}{2000000q}$ nearly, or, more accurately, $r = \frac{.0000045pv^2q^{\frac{1}{8}}}{q}$. These calculations are compared with the whole of DUBUAT'S experiments in the following table.

Table of Experiments on the Resistance occasioned by Flexure.

p	q	v^2	r	B.	Y. 1	Y. 2	p	q	v^2	r	B.	Y. 1	Y. 2
288	3.22	15030	4.75		6.71	6.98	288	3.22	3415	1.50	1.57	1.52	1.58
		11330	3.50		5.06	5.26	144			.75	.78	.76	.79
		7199	2.33		3.21	3.34	72			.37	.39	.38	.39
		3510	1.08		1.56	1.62	196.5	6.12		.75	.78	.55	.62
216		7216	2.49	2.49	2.42	2.52	112.5	.53		1.50		3.63	3.00
144			1.50	1.66	1.61	1.67	720	3.22	5125	5.90	5.90	5.72	5.95
72			.75	.83	.80	.83	288		3458	1.64	1.59	1.54	1.60
196.5	6.12		1.50	1.66	1.16	1.31			860	.41	.40	.38	.40
147.4			1.12	1.24	.87	.98	——		821	.39	.38	.37	.38
98.3			.75	.83	.58	.65	288	4.10	3448	1.33		1.21	1.30
49.1			.37	.41	.29	.33			7449	2.90		2.59	2.78
112.5	.53		6.00		7.68	6.36	294.8	9.9 }					
99			5.90		6.74	5.60	360	4.1 }		8.64		8.08	8.62
288	3.22	3415	1.50	1.57	1.52	1.58	112.5	1.1 }					

In the last three experiments, the diameter of the pipe was two inches. The radius of curvature is not ascertained within the tenth of an inch, as DUBUAT has not mentioned the thickness of the pipes. The mean error of his formula in fifteen experiments, and of mine in twenty, is $\frac{1}{25}$ of the whole.

III. *Of the Propagation of an Impulse through an elastic Tube.*

The same reasoning, that is employed for determining the velocity of an impulse, transmitted through an elastic solid or fluid body, is also applicable to the case of an incompressible fluid contained in an elastic pipe; the magnitude of the modulus being properly determined, according to the excess of pressure which any additional tension of the pipe is capable of producing; its height being such, as to produce a tension, which is to any small increase of tension produced by the approach of two sections of the fluid in the pipe, as their distance to its decrement: for in this case the forces concerned are precisely similar to those which are employed in the transmission of an impulse through a column of air enclosed in a tube, or through an elastic solid. If the nature of the pipe be such, that its elastic force varies as the excess of its circumference or diameter above the natural extent, which is nearly the usual constitution of elastic bodies, it may be shown that there is a certain finite height which will cause an infinite extension, and that the height of the modulus of elasticity, for each point, is equal to half its height above the base of this imaginary column; which may therefore be called with propriety the modular column of the pipe: consequently the velocity of an impulse will be at every point equal to half of that which is due to the height of the point above the base; and the velocity of an impulse ascending through the pipe being every where half as great as that of a body falling through the corresponding point in the modular column, the whole time of ascent will be precisely twice as great as that of the descent of the falling body; and in the same manner if the pipe be inclined, the motion of the impulse may be compared with that of a body descending or ascending freely along an inclined plane.

These propositions may be thus demonstrated: let a be the diameter of the pipe in its most natural state, and let this diameter be increased to b by the pressure of the column c, the tube being so constituted that the tension may vary as the force. Then the relative force of the column c is represented by bc, since its efficacy increases, according to the laws of hydrostatics, in the ratio of the diameter of the tube; and this force must be equal, in a state of equilibrium, to the tension arising from the change from a to b, that is, to $b - a$; consequently the height c varies as $\frac{b-a}{b}$; and if the tube be enlarged to any diameter x, the corresponding pressure required to distend it will be expressed by a height of the column equal to $\left(1 - \frac{a}{x}\right) \cdot \frac{bc}{b-a}$, since $\frac{b-a}{b} : c :: \frac{x-a}{x} : \left(1 - \frac{a}{x}\right) \frac{bc}{b-a}$. Now if the diameter be enlarged in such a degree, that the length of a certain portion of its contents may be contracted in the ratio $1 : 1 - r$, r being very small, then the enlargement will be in the ratio $1 : 1 + \frac{r}{2}$, that is, x' will be $\frac{rx}{2}$; but the increment of the force, or of the height, is $\frac{ax'}{xx} \cdot \frac{bc}{b-a}$, which will become $\frac{ar}{2x} \cdot \frac{bc}{b-a}$. Now in a tube filled with an elastic fluid, the height being b, the force in similar circumstances would be rb, and if we make $h = \frac{a}{2x} \cdot \frac{bc}{b-a}$, the velocity of the propagation of an impulse will be the same in both cases, and will be equal to the velocity of a body which has fallen through the height $\frac{1}{2}\,b$. Supposing x infinite, the height capable of producing the necessary pressure becomes $\frac{bc}{b-a}$, which may be called g, and for every other value of x this height is $\left(1-\frac{a}{x}\right)g$, or $g-\frac{ag}{x}$, or, since b becomes $\frac{ag}{2x}$, $g-2b$, so that b is always equal to half the difference between g and the actual height of the column above the given point, or to half the height of the point above the base of the column.

If two values of x, with their corresponding heights, are given, as b and x, corresponding to c and d, and it is required to find a; we have $\frac{b-a}{b} : c :: \frac{x-a}{x} : d$, $dbx - dax = cbx - cba$, and $a = \frac{dbx-cbx}{dx-cb}$, or $\frac{b}{a} = \frac{dx-cb}{dx-cx}$. Thus if the height equivalent to the tension vary in the ratio of any power m of the diameter, so that, n being a small quantity, $x = b\,(1+n)$ and $d = c\,(1+mn)$, $\frac{b}{a} = \frac{bc\,((1+n).(1+mn)-1)}{bc\,((1+n).(1+mn-(1+n))} = \frac{mn+n}{mn}$, since the square of n is evanescent, and $\frac{b}{a} = \frac{m+1}{m}$. For example, if $m = 4$, $\frac{b}{a} = \frac{5}{4}$, and if $m = 2$, $b : a :: 3 : 2$.

IV. *Of the Magnitude of a diverging Pulsation at different Points.*

The demonstrations of Euler, Lagrange, and Bernoulli, respecting the propagation of sound, have determined that the velocity of the actual motion of the individual particles of an elastic fluid, when an impulse is transmitted through a conical pipe, or diverges spherically from a centre, varies in the simple inverse ratio of the distance from the vertex or centre, or in the inverse subduplicate ratio of the number of particles affected, as might naturally be inferred from the general law of the preservation of the ascending force or impetus, in all cases of the communication of motion between elastic bodies, or the particles of fluids of any kind. There is also another way of considering the subject, by which a similar conclusion may be formed respecting waves diverging from or converging to a centre. Suppose a straight wave to be reflected backwards and forwards in succession, by two vertical surfaces, perpendicular to the direction of its motion; it is evident that in this and every other case of such reflections, the pressure against the opposite surfaces must be equal, otherwise the centre of inertia of the whole system of bodies concerned would be displaced by their mutual actions, which is contrary to the general laws of the properties of the centre of inertia. Now if, instead of one of the surfaces, we substitute two others, converging in a very acute angle, the wave will be elevated higher and higher as it approaches the angle: and if its height be supposed to be every where in the inverse subduplicate ratio of the distance of the converging surfaces, the magnitude of the pressure, reduced to the direction of the motion, will be precisely equal to that of the pressure on the single opposite surface, which will not happen if the elevation vary inversely in the simple ratio of the distance, or in that of any other power than its square root. This mode of considering the subject affords us therefore an additional reason for asserting, that in all transmissions of impulses through elastic bodies, or through gravitating fluids, the intensity of the impulse varies inversely in the subduplicate ratio of the extent of the parts affected at the same time; and the same reasoning may without doubt be applied to the case of an elastic tube.

There is however a very singular exception, in the case of waves crossing each other, to the general law of the preservation of ascending force, which appears to be almost sufficient to set aside the universal application of this law to the motions of fluids. It is confessedly demonstrable that each of two waves, crossing each other in any direction, will preserve its motion and its elevation with respect to the surface of the fluid affected by the other wave, in the same manner as if that surface were plane: and, when the waves cross each other nearly in the same direction, both the height and the actual velocity of the particles being doubled, it is obvious that the ascending force or impetus is also doubled, since the bulk of the matter concerned is only halved, while the square of the velocity is quadrupled; and supposing the double wave to be stopped by an obstacle, its magnitude, at the moment of the greatest elevation, will be twice as great as that of a single wave in similar circumstances, and the height, as well as the quantity of matter, will be doubled, so that either the actual or the potential height of the centre of gravity of the fluid seems to be essentially altered, whenever such an interference of waves takes place. This difficulty deserves the attentive consideration of those who shall attempt to investigate either the most refined parts of hydraulics, or the metaphysical principles of the laws of motion.

V. *Of the Effect of a Contraction, advancing through a Canal.*

IF we suppose the end of a rectangular horizontal canal, partly filled with water, to advance with a given velocity, less than that with which a wave naturally moves on the surface of the water, it may be shown that a certain portion of the water will be carried forwards, with a surface nearly horizontal, and that the extent of this portion will be determined, very nearly, by the difference of the spaces described, in any given time, by a wave, moving on the surface thus elevated, and by the moveable end of the canal. The form of the anterior termination of this elevated portion, or wave, may vary, according to the degrees by which the motion may be supposed to have commenced; but whatever this form may be, it will cause an accelerative force, which is sufficient to impart successively to the portions of the fluid, along which it passes, a velocity equal to that of the moveable end, so that the elevated surface of the parts in motion may remain nearly horizontal: and this proposition will be the more accurately true, the smaller the velocity of the moveable end may be. For, calling this velocity v, the original depth a, the increased depth x, and the velocity of the anterior part of the wave y, we have, on the supposition that the extent of the wave is already become considerable, $x = \frac{ay}{y \mp v}$, taking the negative or positive sign according to the direction of the motion of the end; since the quantity of fluid, which before occupied a length expressed by y, now occupies the length $y \mp v$; and putting $a \sim x = z$, $z = \frac{av}{y \mp v}$. The direction of the surface of the margin of the wave is indifferent to the calculation, and it is most convenient to suppose its inclination equal to half a right angle, so that the accelerating force, acting on any thin transverse vertical lamina, may be equal to its weight: then the velocity y must be such, that while the inclined margin of the wave passes by each lamina, the lamina may acquire the velocity v by a force equal to its own weight; consequently the time of its passage must be equal to that in which a body acquires the velocity v, in falling through a height b, corresponding to that velocity; and this time is expressed by $\frac{2b}{v}$; but the space described by the margin of the wave is not exactly z, because the lamina in question has moved horizontally during its acceleration, through a space which must be equal to b; the distance actually described will therefore be $z \pm b$, and we have $\frac{z \pm b}{y} = \frac{2b}{v}$, $z \pm b = \frac{2by}{v}$, $av \pm by - bv = \frac{2byy}{v} \mp 2by$, $y^2 \mp \frac{3}{2} vy = \frac{av^2}{2b} - \frac{v^2}{2}$, $(y \mp \frac{3}{4} v)^2 = \frac{av^2}{2b} + \frac{v^2}{16}$; but, m being the proper coefficient, $v = m \sqrt{b}$, and $v^2 = m^2 b$, $\frac{av^2}{2b} + \frac{v^2}{16} = m^2 \left(\frac{a}{2} + \frac{b}{16}\right)$, $y = m \sqrt{\left(\frac{a}{2} + \frac{b}{16}\right)} \pm \frac{3}{4} v$, and $y \mp v = m \sqrt{\left(\frac{a}{2} + \frac{b}{16}\right)} \mp \frac{1}{4} v$. But when v is small, we may take $y \mp v$ nearly $m \sqrt{\frac{a}{2}}$, and $z = \frac{ma \sqrt{b}}{m \sqrt{(\frac{1}{2} a)}} = \sqrt{(2ab)}$, and $x = a \pm \sqrt{(2ab)}$, while the height of a fluid, in which the velocity would be y, is nearly $a \pm \frac{3}{2} \sqrt{(2ab)}$: consequently, when the velocity v is at all considerable, y must be somewhat greater than the velocity of a wave moving on the surface of the elevated fluid; and probably the surface of the elevated portion will not in this case be perfectly horizontal; but where v is small, y may be taken, without material error, $m \surd \frac{x}{2}$, or even $m \surd \frac{a}{2}$, which is the velocity of every small wave. The coefficient m is here assumed the same for the motion of a wave, as for the discharge through an aperture, and I have reason from observation to think this estimation sufficiently correct.

Supposing now the moveable end of the canal to remain open at the lower part as far as the height c, then the excess of pressure, occasioned by the elevation before it, and the depression behind, will cause the fluid, immediately below the moveable plane, to flow backwards, with the velocity determined by the height, which is the difference between the levels; and the quantity thus flowing back, together with that which is contained in the moveable elevation, must be equal to the whole quantity displaced. But the depression, behind the moveable body, must vary according to the circumstances of the canal, whether it be supposed to end abruptly at the part from which the motion begins, or to be continued backwards without limit: in the first case, the elevation z will be to the depression as v to $y - v$, the length of the same portion of the fluid being varied inversely in that ratio; in the second case, the proportion will be as $y + v$ to $y - v$: and the difference of the levels will be $z + z\frac{y-v}{v} = \frac{zy}{v}$, or secondly $z + z\frac{y-v}{y+v} = \frac{2zy}{y+v}$; and first, $m \surd \frac{zy}{v} c + (y - v) z = (a-c) v$; but, since y is here considered as equal to $m \surd \frac{a}{1}$, putting $\surd \frac{a}{2} - \surd b = d$, $y - v = md$, and, calling $a - c$, e, $m \surd \frac{zy}{v} c + mdz = me \surd b$, $\surd \frac{zy}{v} c + dz = e \surd b$, $c^2 \frac{zy}{v} = e^2 b + d^2 z^2 - 2dze \surd b$, $z^2 - \left(\frac{c^2 y}{d^2 v} + \frac{2e \surd b}{d}\right) z = - \frac{e^2 b}{d^2}$, and, calling $\frac{c^2 y}{2d^2 v} +$

$\frac{e\sqrt{b}}{d}$, f, $z = f - \sqrt{\left(f^2 - \frac{e^2 b}{d^2}\right)}$: and in the same manner f is found, for the second case, equal to $\frac{c^2 y}{d^2 (y + v)} + \frac{e\sqrt{b}}{d}$. For example, suppose the height a 2 feet, $b = \frac{1}{4}$, $c = 1$, and consequently $e = 1$, then d becomes $\frac{1}{2}$, $v = 4$, and $y = 8$; and in the first case $z = .1$, and in the second $z = .14$.

If v, the velocity of the obstacle, were great in comparison with $m\sqrt{\frac{a}{2}}$, the velocity of a wave, and the space c below the obstacle were small, the anterior part of the elevation would advance with a velocity considerably greater than the natural velocity of the wave: but if the space below the obstacle bore a considerable proportion to the whole height, the elevation z would be very small, since a moderate pressure would cause the fluid to flow back, with a sufficient velocity, to exhaust the greatest part of the accumulation, which would otherwise take place. Hence the elevation must always be less than that which is determined by the equation $m\sqrt{zc} = ev$, and z is at most equal to $\left(\frac{ev}{mc}\right)^2 = \frac{e^2}{c^2} b$; but since the velocity of the anterior margin of the wave can never materially exceed $m\sqrt{\frac{x}{2}}$, especially when z is small, and $\sqrt{\frac{x}{2}}$ being in this case nearly $\sqrt{\frac{a}{2}} + \frac{e^2}{2\sqrt{(\frac{1}{2}a)c^2}} b$, $m\sqrt{\frac{x}{2}} - m\sqrt{b} = m\left(\sqrt{\frac{a}{2}} + \frac{e^2 b}{\sqrt{(2a)c^2}} - \sqrt{b}\right)$ which, multiplied by z, shows the utmost quantity of the fluid that can be supposed to be carried before the obstacle. Supposing $b = \frac{1}{2} a$, this quantity becomes $m\sqrt{\frac{a}{2}} \cdot \frac{e^4}{c^4} \cdot \frac{a}{4}$; and if $\frac{e}{c}$ be, for example, $\frac{1}{10}$, it will be expressed by $\frac{1}{40000} av$, while the whole quantity of the fluid left behind.

A similar mode of reasoning may be applied to other cases of the propagation of impulses, in particular to that of a contraction moving along an elastic pipe. In this case, an increase of the diameter does not increase the velocity of the transmission of an impulse; and when the velocity of the contraction approaches to the natural velocity of an impulse, the quantity of fluid protruded must, if possible, be still smaller than in an open canal; that is, it must be absolutely inconsiderable, unless the contraction be very great in comparison with the diameter of the pipe, even if its extent be such as to occasion a friction which may materially impede the retrograde motion of the fluid. The application of this theory to the motion of the blood in the arteries is very obvious, and I shall enlarge more on the subject when I have the honour of laying before the Society the Croonian Lecture for the present year.

The resistance, opposed to the motion of a floating body, might in some cases be calculated in a similar manner: but the principal part of this resistance appears to be usually derived from a cause which is here neglected; that is, the force required to produce the ascending, descending, or lateral motions of the particles, which are turned aside to make way for the moving body; while in this calculation their direct and retrograde motions only are considered.

The same mode of considering the motion of a vertical lamina may also be employed for determining the velocity of a wave of finite magnitude. Let the depth of the fluid be a, and suppose the section of the wave to be an isosceles triangle, of which the height is b, and half the breadth c : then the force urging any thin vertical lamina in a horizontal direction will be to its weight as b to c; and the space d, through which it moves horizontally, while half the wave passes it, will be such that $(c-d).(a+\frac{1}{2}b) = ac$, when $c\ e\ d = \frac{bc}{2a+b}$. But the final velocity in this space is the same as is due to a height equal to the space, reduced in the ratio of the force to the weight, that is, to the height $\frac{bb}{2a+b}$, and half this velocity is $\frac{1}{2} m \surd \left(\frac{bb}{2a+b}\right)$, which is the mean velocity of the lamina. In the mean time the wave describes the space $c+d$, and its velocity is greater than that of the lamina in the ratio of $\frac{c}{d} + 1$ to 1, that is $\frac{2a+b}{b} + 1$ or $\frac{2a}{b} + 2$ to 1, becoming $m \left(\frac{a}{b} + 1\right) \frac{b}{\surd(2a+b)} = m \frac{a+b}{\surd(2a+b)}$; which, when b vanishes, becomes $m \surd \frac{a}{2}$, as in LAGRANGE's theorem, and, when b is small, $m \left(\surd \frac{a}{2} + \frac{3}{4} \frac{b}{\surd(2a+b)}\right)$, or $m \frac{a+\frac{3}{4}b}{\surd(2a)}$; but if a were small, it would approach to $m \surd b$, the velocity due to the whole height of the wave.